Answers to Coursebook questions – Chapter E6

1 It is a spiral galaxy.

2 It is a spiral galaxy some 100 000 light years across, with a central spherical bulge. The thickness at the outer edges is of order 2000 light years.

3 The distant galaxies move away from earth with a speed that is proportional to their distance from earth. This is evidence for the Big Bang because in the past these galaxies must have been very close to each other, i.e. in the very distant past the separation between particles must have been very small.

4 No. These are nearby galaxies which show a blueshift because the mutual gravitational attraction between them and our Milky Way makes them move toward us.

5 Taking the Hubble constant to be $H = 72\text{ km s}^{-1}\text{ Mpc}^{-1}$ and Hubble's law,

we find $v = Hd \Rightarrow d = \frac{v}{H} = \frac{500}{72} \approx 7\ \text{Mpc}$.

6 No, because these are nearby galaxies whose motion is much more affected by their mutual gravitational attraction rather than by the cosmic expansion.

7 **a** $$\frac{v}{c} = \frac{\Delta\lambda}{\lambda_0} \Rightarrow v = \frac{c\Delta\lambda}{\lambda_0} = \frac{3\times10^8\times(658.9-656.3)}{656.3} \approx 1.2\times10^6\ \text{m s}^{-1}.$$

$v = Hd \Rightarrow d = \frac{v}{H} = \frac{1.2\times10^6}{72\times10^3} \approx 16.5\ \text{Mpc}$. We have expressed the Hubble constant as $H = 72\times10^3\ \text{m s}^{-1}\ \text{Mpc}^{-1}$ so that the distance comes out in Mpc.

b $$\frac{v}{c} = \frac{\Delta\lambda}{\lambda_0} \Rightarrow v = \frac{c\Delta\lambda}{\lambda_0} = \frac{3\times10^8\times(689.1-656.3)}{656.3} \approx 1.5\times10^7\ \text{m s}^{-1}.$$

$$v = Hd \Rightarrow d = \frac{v}{H} = \frac{1.5\times10^7}{72\times10^3} \approx 208\ \text{Mpc}.$$

c $$\frac{v}{c} = \frac{\Delta\lambda}{\lambda_0} \Rightarrow v = \frac{c\Delta\lambda}{\lambda_0} = \frac{3\times10^8\times(704.9-656.3)}{656.3} \approx 2.2\times10^7\ \text{m s}^{-1}.$$

$$v = Hd \Rightarrow d = \frac{v}{H} = \frac{2.2\times10^7}{72\times10^3} \approx 306\ \text{Mpc}.$$

d $$\frac{v}{c} = \frac{\Delta\lambda}{\lambda_0} \Rightarrow v = \frac{c\Delta\lambda}{\lambda_0} = \frac{3\times10^8\times(741.6-656.3)}{656.3} \approx 3.9\times10^7\ \text{m s}^{-1}.$$

$$v = Hd \Rightarrow d = \frac{v}{H} = \frac{3.9\times10^7}{72\times10^3} \approx 542\ \text{Mpc}.$$

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e $\frac{v}{c}=\frac{\Delta\lambda}{\lambda_0} \Rightarrow v=\frac{c\Delta\lambda}{\lambda_0}=\frac{3\times10^8\times(789.7-656.3)}{656.3}\approx 6.1\times10^7\ \text{m s}^{-1}$.

$v=Hd \Rightarrow d=\frac{v}{H}=\frac{6.1\times10^7}{72\times10^3}\approx 847\ \text{Mpc}$.

8 **a** $v=Hd \Rightarrow d=\frac{v}{H}=\frac{3\times10^5}{72}\approx 4200\ \text{Mpc}\approx 10^{26}\ \text{m}$.

b This is the edge of the observable universe – we cannot observe anything beyond this distance.

c This is not in violation of relativity. This is not the speed of any inertial frame of reference. It is a speed due to the stretching of space in between galaxies. This speed cannot be used to send a signal.

9 **a** The redshift is $\frac{\Delta\lambda}{\lambda_0}=\frac{5.3\times10^{-7}-4.5\times10^{-7}}{4.5\times10^{-7}}=0.178\approx0.18$.

b $\frac{\Delta\lambda}{\lambda_0}=\frac{v}{\text{c}}$ so $v=0.178\text{c}\approx5.3\times10^4\ \text{km s}^{-1}$.

c From Hubble's law, $v=Hd$, we have that $d=\frac{v}{H}=\frac{5.3\times10^4\ \text{km s}^{-1}}{72\ \text{km s}^{-1}\ \text{Mpc}^{-1}}\approx 740\ \text{Mpc}$.

10 At the time of the Big Bang the distance between any two objects was zero.
A galaxy that today is at a distance of *d* travelled this distance in time *T*, the age of the universe today, and so $v=\frac{d}{T}$.
The present speed of recession of the galaxy is $v=Hd$.
Assuming that the galaxy had this speed throughout, then $\frac{d}{T}=Hd$, i.e. $T=\frac{1}{H}$.
With $H=500\ \text{km s}^{-1}\ \text{Mpc}^{-1}$,
$T=\frac{1}{500\ \text{km s}^{-1}\ \text{Mpc}^{-1}}=\frac{1}{500\times10^3}10^6\times3.26\times9.46\times10^{15}\ \ \text{s}$,
i.e. $T=6.2\times10^{19}\ \ \text{s}$ or 2 billion years. (The earth is older than this estimate.)

11 The estimate of the age of the universe as $T=\frac{1}{H}$ is based on the assumption that the galaxies have been moving away from earth at their present speed.
The speed in the past was slower, however, and so the age of the universe must be less than $T=\frac{1}{H}$.

12 As explained on page 538 in *Physics for the IB Diploma*, the fact that the speed of recession is proportional to the distance from earth implies that any other observer, anywhere else in the universe, would reach the same conclusion, i.e. that he/she is at the centre of the expansion. Thus, there is no centre of expansion.

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13 **a** The distance is $r = R\theta$ and so $v = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t}\theta = \frac{\mathrm{d}R}{\mathrm{d}t}\frac{r}{R}$.

Hence $v = Hr$ with $H = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}$.

b If $R = kt$, then $H = \frac{1}{R}\frac{\mathrm{d}(kt)}{\mathrm{d}t} = \frac{k}{kt} = \frac{1}{t}$.

c We must require that $H = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \text{const} = \mathrm{c}$, i.e.

$$\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \mathrm{c}$$
$$\frac{\mathrm{d}R}{R} = \mathrm{c}\,\mathrm{d}t$$
$$\ln R = \mathrm{c}t + \ln R_0$$
$$R = R_0 \mathrm{e}^{\mathrm{c}t}$$

i.e. an exponential increase with time.

14 The time (about 3 minutes after the Big Bang) when the temperature dropped sufficiently for light nuclei to form.

15 Because the temperature was still so high that the average thermal energy of electrons prevented them from staying in bound states in atoms.

16 See **Q14**.

17 The very early universe has a very small excess of particles over antiparticles. Collisions between particles and antiparticles annihilated the matter and antimatter into photons. The reverse process also occurred, though with photons materializing into particle–antiparticle pairs, resulting in an equilibrium situation. As the temperature fell below the threshold for producing the particle–antiparticle pairs, this process stopped, leaving only the annihilation of the particles and the antiparticles. This left the universe with only the small amount of excess particles it had originally.

18 At the time of nucleosynthesis (formation of light nuclei) at about 3 minutes after the Big Bang, there were 14 protons for every 2 neutrons in the universe. These combined to form one nucleus of helium (2 protons and 2 neutrons) for every 12 nuclei of hydrogen (12 protons). The mass of the helium nucleus is about 4 u and that of the 8 hydrogen nuclei is about 12 u. Thus the proportion of helium relative to hydrogen is about 25%. The fact that this is observed in stars and galaxies is strong evidence for the Big Bang, which predicts the original ratio of 14 protons to 2 neutrons.

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19 Assume a spherical universe of radius R. Then the density is $\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$.

So $\frac{d\rho}{dt} = 0 = \frac{3}{4\pi R^3}\frac{dM}{dt} - \frac{9M}{4\pi R^4}\frac{dR}{dt}$, i.e. $\frac{dM}{dt} = \frac{3M}{R}\frac{dR}{dt}$.

In other words, $\frac{dM}{dt} = 4\pi\rho R^2 \frac{dR}{dt}$.

From Hubble's law, $v = \frac{dR}{dt} = HR$ and so $\frac{dM}{dt} = 4\pi\rho R^3 H$. Finally, $\frac{1}{V}\frac{dM}{dt} = 3\rho H$.

Numerically, $\frac{1}{V}\frac{dM}{dt} = 3\times10^{-26}\times\frac{72\times10^3}{10^6\times3.26\times9.46\times10^{15}}$ kg m^{-3} s^{-1}.

This gives, $\frac{1}{V}\frac{dM}{dt} = 7\times10^{-44}$ kg m^{-3} s^{-1}. This is very small!

20 This exercise is for those students taking higher-level maths. Everything here (except for the last bit involving hyperbolic functions) is perfectly within their sphere of knowledge of calculus – it is a good opportunity to apply what they have learned in math in a concrete physical setting. You cannot learn physics without using mathematics!

From the equation of motion $m\frac{dv}{dt} = -\frac{4\pi\rho r^3}{3}\frac{Gm}{r^2}$, we get the simplified form $\frac{dv}{dt} = -\frac{4\pi G\rho}{3}r$. (The minus sign is needed as the force is directed towards the centre of the attracting body, i.e. opposite to the direction in which the variable r is increasing.)

Now by Hubble's law, $v = Hr = HRr_0$ and $v = \frac{dr}{dt} = \frac{d(Rr_0)}{dt} = \frac{dR}{dt}r_0$. Thus, $HR = \frac{dR}{dt}$.

This means that we can write $v = \frac{dR}{dt}r_0$ without reference to the Hubble constant.

Putting this in the equation of motion gives:

$$\frac{d}{dt}\left(\frac{dR}{dt}r_0\right) = -\frac{4\pi G\rho}{3}Rr_0$$

$$\frac{d^2R}{dt^2} + \frac{4\pi G\rho}{3}R = 0$$

In these equations the variables r_0 and ρ_0 refer to the 'radius' and density of the universe at a particular time which may be taken to be the present values.

Because of conservation of mass we have that $\rho\frac{4\pi r^3}{3} = \rho_0\frac{4\pi r_0^3}{3}$ (because each expression gives the mass of the universe at different times and the mass is the same) and so $\rho\frac{4\pi R^3 r_0^3}{3} = \rho_0\frac{4\pi r_0^3}{3} \Rightarrow \rho = \frac{\rho_0}{R^3}$.

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Hence, back into the equation:

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{4\pi GR}{3}\frac{\rho_0}{R^3}=0$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{4\pi G\rho_0}{3}\frac{1}{R^2}=0$$

$$2\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{C}{R^2}=0$$

where the constant is given by $C=\frac{8\pi G\rho_0}{3}$. We must now solve $2\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{C}{R^2}=0$.

Since $2\frac{\mathrm{d}^2R}{\mathrm{d}t^2}=2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}R}{\mathrm{d}t}=2\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2$ it follows by integration term by term with respect to R that

$$\int\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\mathrm{d}R+\int\frac{C}{R^2}\mathrm{d}R=0$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}+k=0$$

where k is a constant of integration. We have derived the Friedmann equation:

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}+k=0.$$

If $k=0$, we must solve $\frac{\mathrm{d}R}{\mathrm{d}t}=\sqrt{\frac{C}{R}}\Rightarrow\sqrt{R}\,\mathrm{d}R=\sqrt{C}\,\mathrm{d}t$.

Integrating this separable equation gives $\frac{2R^{3/2}}{3}=\sqrt{C}t$, i.e. $R=\sqrt[3]{\frac{9C}{4}}\,t^{2/3}$.

This shows that the time dependence of the scale factor is $R\propto t^{2/3}$.

The rate of change of the scale factor with time is then

$$\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{3\sqrt{C}}{2}\frac{\mathrm{d}}{\mathrm{d}t}t^{2/3}=\frac{3\sqrt{C}}{2}\frac{2}{3}t^{-1/3}=\sqrt{C}t^{-1/3}.$$

Thus as $t\to\infty$, $\frac{\mathrm{d}R}{\mathrm{d}t}\to 0$, the characteristic of the flat universe in which the expansion continues to infinity but at a rate that becomes zero at infinity.

Recall that $HR=\frac{\mathrm{d}R}{\mathrm{d}t}$, and so we can now derive an expression for the Hubble 'constant': $H=\frac{1}{R}\sqrt{C}t^{-1/3}=\frac{2}{3\sqrt{C}t^{2/3}}\sqrt{C}t^{-1/3}=\frac{2}{3t}$.

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In the text (see page 538 in *Physics for the IB Diploma*), it was shown under the approximation of a constant rate of expansion at the present rate, the Hubble constant is given by $H=\frac{1}{t}$. The expression derived here corrects this assumption and, as we can see, the change is just by a number of order 1, so the estimate of the age of the universe based on $H=\frac{1}{t}$ is not entirely unreasonable and serves as an upper bound to the age of the universe.

The Friedmann equation, $\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}+k=0$, shows that if we want $\frac{\mathrm{d}R}{\mathrm{d}t}=0$ then $k=\frac{C}{R}>0$. In other words, for the expansion to come to a halt, the parameter k must be positive. So we must solve this equation for $k>0$.

Because the equation involves the term $\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2$ it will be necessary to be careful about the sign of k. We can verify that the solution for $k>0$ is (see sketch at the end of this exercise)

$$R(\theta)=\frac{C}{2k}(1-\cos\theta) \qquad \frac{\mathrm{d}R}{\mathrm{d}\theta}=\frac{C}{2k}\sin\theta$$

$$t(\theta)=\frac{C}{2k^{3/2}}(\theta-\sin\theta) \qquad \frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{C}{2k^{3/2}}(1-\cos\theta)$$

so that

$$\begin{aligned}
\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}+k &= \frac{k\sin^2\theta}{(1-\cos\theta)^2}-\frac{2k}{(1-\cos\theta)}+k \\
&= k\frac{\sin^2\theta-2(1-\cos\theta)}{(1-\cos\theta)^2}+k \\
&= k\frac{1-\cos^2\theta-2+2\cos\theta}{(1-\cos\theta)^2}+k \\
&= -k\frac{\cos^2\theta-2\cos\theta+1}{(1-\cos\theta)^2}+k \\
&= -k\frac{(1-\cos\theta)^2}{(1-\cos\theta)^2}+k \\
&= -k+k \\
&= 0
\end{aligned}$$

The largest value of R is obtained when $\cos\theta=-1$ and equals $R=\frac{C}{k}(1-(-1))=\frac{2C}{k}$.

This happens at a time given by: $\cos\theta=-1\Rightarrow\theta=\pi$

and so $t=\frac{C}{2k^{3/2}}(\pi-\sin\pi)=\frac{C\pi}{2k^{3/2}}$.

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The next time the scale factor becomes zero, the 'big crunch' is:

$R(\theta)=\dfrac{C}{2k}(1-\cos\theta)=0\Rightarrow\cos\theta=1\Rightarrow\theta=2\pi$.

Thus, $t=\dfrac{C}{2k^{3/2}}(2\pi-\sin 2\pi)=\dfrac{C\pi}{k^{3/2}}$.

Small t means small θ and so by the small angle approximations:

$$R(\theta)=\frac{C}{2k}(1-\cos\theta)\approx\frac{C}{2k}(1-(1-\frac{\theta^2}{2}))=\frac{C\theta^2}{4k}$$

$$t(\theta)=\frac{C}{2k^{3/2}}(\theta-\sin\theta)\approx\frac{C}{2k^{3/2}}(\theta-(\theta-\frac{\theta^3}{6})=\frac{C\theta^3}{12k^{3/2}}$$

This means that $\theta=\sqrt[3]{\dfrac{12k^{3/2}}{C}}t^{1/3}$

and so

$$R=\frac{C}{4k}\left(\frac{12k^{3/2}}{C}\right)^{2/3}t^{2/3}$$
$$=\sqrt[3]{\frac{9C}{4}}\,t^{2/3}$$

The dependence on k in fact cancels out, and the dependence of R on time is exactly identical to the behaviour for $k=0$.

We expect the rate of change to be largest for small times, so we use the small time approximation to find:

rate of change of R with time $\dfrac{\mathrm{d}R}{\mathrm{d}t}=\sqrt[3]{\dfrac{9C}{4}}\dfrac{\mathrm{d}}{\mathrm{d}t}t^{2/3}=\sqrt[3]{\dfrac{9C}{4}}\dfrac{2}{3t^{1/3}}$.

This is largest for very small times, as expected.

For $k<0$ the analysis is similar and leads to the solutions

$$R(\theta)=\frac{C}{2|k|}(\cosh\theta-1)$$

$$t(\theta)=\frac{C}{2|k|^{3/2}}(\sinh\theta-\theta)$$

where $\cosh\theta=\dfrac{\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}}{2}$ and $\sinh\theta=\dfrac{e^{\theta}-e^{-\theta}}{2}$.

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As a check,

$$\frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta}=\frac{C}{2|k|}\sinh\theta \text{ and } \frac{\mathrm{d}t(\theta)}{\mathrm{d}\theta}=\frac{C}{2|k|^{3/2}}(\cosh\theta-1)$$

so that $\sinh\theta=\frac{2|k|}{C}\frac{\mathrm{d}R}{\mathrm{d}t}$ and $\cosh\theta-1=\frac{2|k|^{3/2}}{C}\frac{\mathrm{d}t}{\mathrm{d}\theta}$.

Then (we will use the identity $\cosh^2\theta-\sinh^2\theta=1$)

$$\begin{aligned}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}-|k| &= \frac{|k|\sinh^2\theta}{(\cosh\theta-1)^2}-\frac{2|k|(\cosh\theta-1)}{(\cosh\theta-1)^2}-|k| \\ &= |k|\frac{\cosh^2\theta+1-2\cosh\theta}{(\cosh\theta-1)^2}-|k| \\ &= |k|\frac{(\cosh\theta-1)^2}{(\cosh\theta-1)^2}-|k| \\ &= 0\end{aligned}$$

For small times, $\cosh x-1\approx\frac{x^2}{2}$ and $\sinh x-x\approx\frac{x^3}{6}$, so that

$$R(\theta)=\frac{C}{2|k|}(\cosh\theta-1)\approx\frac{C\theta^2}{4|k|}$$

$$t(\theta)=\frac{C}{2|k|^{3/2}}(\sinh\theta-\theta)\approx\frac{C\theta^3}{12|k|^{3/2}}$$

implying that $R=\sqrt[3]{\frac{9C}{4}}\,t^{2/3}$, showing that the small-time behaviour is the same in all cases. This can be understood through the Friedmann equation, $\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2-\frac{C}{R}+k=0$.

For small times, R is very small and so $\frac{1}{R}$ and its rate of change are enormous, whereas the term with k in this equation is irrelevantly small and can be neglected. Hence in all cases we have the same small-time behaviour.

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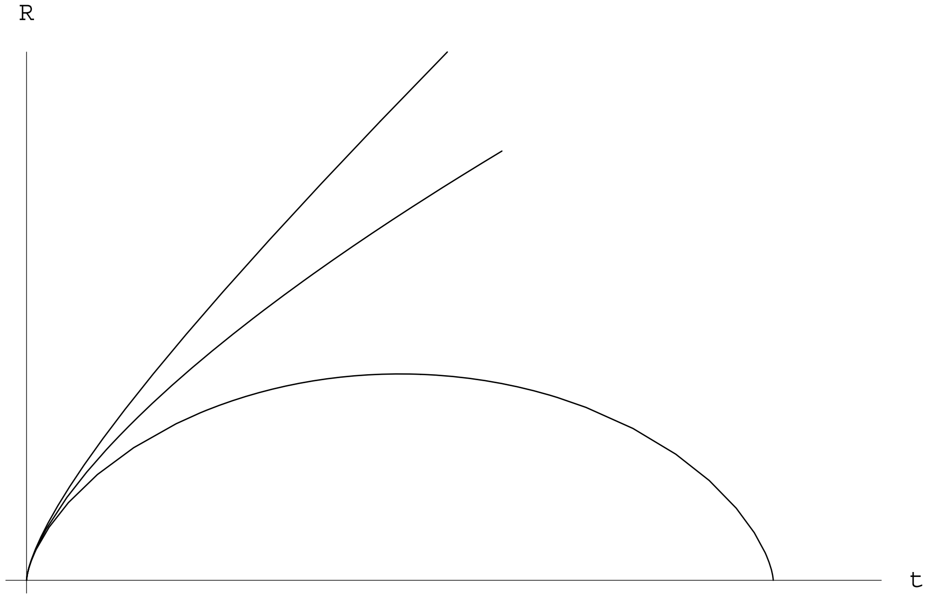
The variation of the scale factor with time in the three cases is shown below. (Notice that the curves have been made to start from a common point – this implies that the present time is at different points on the three graphs because the age of the universe depends on the model chosen.)



It is reasonable to assume that the parameter k in this model is related to the curvature of space: we saw in the text (see pages 517–518 in *Physics for the IB Diploma*) that the three possibilities in this model correspond to three possible geometries of the universe: closed ($k > 0$), open ($k < 0$) and flat ($k = 0$).

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